Applied Combinatorics, Section N2

Final Exam

Name: ______________________________

GTID: ______________________________

Problem	*Points*
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

TOTAL:_________________

Please do show all your work including intermediate steps and also explain in words how you solve a problem. Partial credits are available.

Problem 1. Determine if the following statements are true or false. If it is false, please give a conter-example.

(1) Any tree has at least two leaves.

(2) A graph is Eulerian if and only if every vertex of the graph is even

(3) K_4 is a planar graph

(4) Any edge added to a tree must produce a cycle

(5) A Hamiltonian graph is Eulerian, but an Eulerian graph is not necessarily Hamiltonian.

Problem 2. If $\mathbf{E}[X] = 2$ and $\mathbf{Var}(X) = 4$. Find $\mathbf{E}[X^2 + 2X]$.

Problem 3. Ten identical objects are to be put into three distinct boxes. How many ways are there such that no box is empty and each box contains at most 4 objects?

Problem 4. A commitee of 10 women and 10 men is to be seated at a circular table. In how many ways can the seats be assigned so that no two men are seated next to each other?

Problem 5. Show that

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2}$$

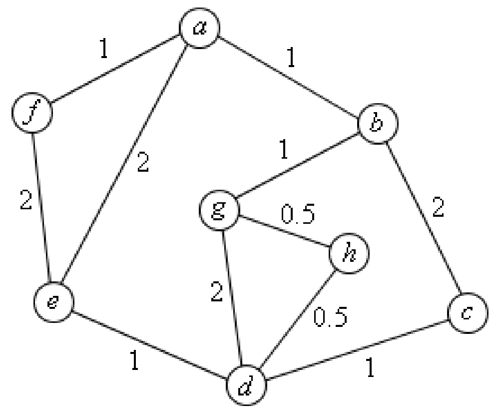
by taking the second derivative of the expansion of $(1 + x)^n$.

Problem 6. What is the remainder of $1986^{715} + 1993^{1212}$ divided by 11?

Problem 7. Use the method of generating function to solve

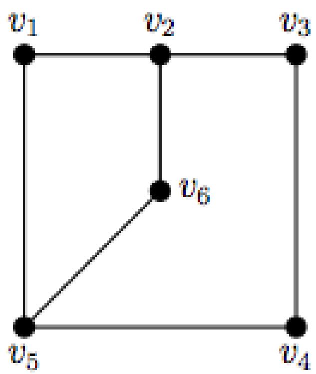
$$a_{n+2} = 5a_{n+1} - 4a_n, \quad a_0 = 2, a_1 = 5.$$

Problem 8. Find the shortest path between vertex f and h in the following graph

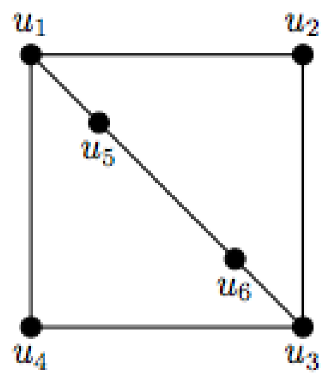


Problem 9. Given the following two graphs G_1 and G_2,

(1) Find the adjacency matrix of G_1;
(2) Explain why G_1 is not Hamiltonian;
(3) Explain why G_2 is not Eulerian;
(4) Show that G_1 and G_2 are isomorphic



G_1



G_2

Problem 10. A gambler starts with an initial fortune of i dollars. On each successive game, the gambler win \$1 with probability 2/3 or loses \$1 with probability 1/3. He will stop if he either accumulates N dollars or loses all his money. We are interested in his probability that he will end up with N dollars.

(1) Let P_i be the probability that the gambler's fortune will reach N instead 0 starting from initial fortune of i dollars. Show by conditioning on the result of the first play that

$$P_i = \frac{2}{3}P_{i+1} + \frac{1}{3}P_{i-1}, \quad 0 \le i \le N$$

and $P_0 = 0, P_N = 1$.

(2) Solve the recurrence relation, i.e. find an explicit formular for P_i.

$$\int_0^1 \int_0^1 xy \, \mathrm{d}x \mathrm{d}y$$