HOMEWORK 1

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly(It is up to you whether to print it double sided or not). Due on Friday, Jan. 20.

Problem 1. (1) Use induction to show that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

(2) Here is a different method to compute the above sum. Notice that

$$(i+1)^4 - i^4 = ai^3 + f(i)$$

where a is a constant and $f(i)$ is a polynomial of degree less than 3. Find a, $f(i)$ and then compute the sum.

Problem 2. It is not hard to see that the number $(2+\sqrt{3})^n$ can be written in the form $a_n + b_n\sqrt{3}$. Show inductively that a_n, b_n satisfies $a_n^2 - 3b_n^2 = 1$.

Problem 3. Show that if $x \neq y$, then the polynomial $x - y$ divides $x^n - y^n$.

Problem 4. Show that for each $n \in \mathbf{N}$, $7^{2n} - 48n - 1$ is a multiple of 2304.

Problem 5. Prove for all integers $n > 1$ the inequality

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Problem 6. Use an inductive argument in jumps of 3 to show that no number of the form $2^n + 1$ is a multiple of 7.

Problem 7. Let $a_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$.

(1) Show that $a_{n+1} = 6a_n - 4a_{n-1}$ by direct calculation.
(2) Show that $2^n \mid a_n$.

Problem 8. Prove inductively that the product of r consecutive integers is divisible by $r!$.

Problem 9. Show that for any $n \in \mathbf{N}$

$$(1+\frac{1}{2^3})(1+\frac{1}{3^3})\cdots(1+\frac{1}{n^3}) \leq 3.$$

Problem 10.

(1) Prove that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, where i is the complex number such that $i^2 = -1$.

(2) Using induction, prove that for all $n \in \mathbf{N}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(3) Verify that $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and compute $(1+i)^{100}$.

Problem 11. A sequence a_n satisfies $a_1 = a_2 = 4$ and $a_{n+1}a_{n-1} = (a_n - 6)(a_n - 12)$ for $n = 2, 3, \ldots$. Show that a_n is a constant, i.e. $a_n = 4$ for any n.

Problem 12. Show that for each natural number N there is an n and appropriate choice of $+$ and $-$ signs (which we write as $\pm$ in short) such that $N = \pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$.(Hint: jump induction)

Problem 13. Show inductively that

(1) $2^n > n^2$ for $n \geq 5$;
(2) $2^n > n^3$ for $n \geq 10$.

Problem 14. For any integer $n > 23$, there exist nonnegative integers x, y such that $n = 7x + 5y$.

Problem 15. Let a_n denote the Fibonacci sequence, i.e. $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$. Prove

(1) $a_n^2 + a_{n+1}^2 = a_{2n+1}$;
(2) $2a_n a_{n+1} + a_{n+1}^2 = a_{2n+2}$.