HOMEWORK 2

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly(It is up to you whether to print it double sided or not). Due on Friday, Jan. 27.

Problem 1. Find a recurrence relation with initial condition that determines each of the following sequences. Notice the answer is not unique.

(1) $2, 4, 16, 256, 65536, \ldots$
(2) $1, 1, 2, 3, 4, 9, 8, 27, \ldots$

Problem 2. Solve the following recurrence relations

(1) $a_n = 5a_{n-1} + 6a_{n-2}, \quad n \geq 2,\ a_0 = 1, a_1 = 3;$
(2) $a_n = 5a_{n-1} + 6a_{n-2}, \quad n \geq 2,\ a_0 = 0, a_1 = 0;$
(3) $a_{n+2} + a_n = 0, \quad n \geq 0,\ a_0 = -1, a_1 = 3;$

Problem 3. Solve $a_{n+2} = 5a_{n+1} + 6a_n + n + 2^n, \quad n \geq 0,\ a_0 = 1, a_1 = 3.$

Problem 4. Solve $a_{n+2}^2 - 5a_{n+1}^2 - 6a_n^2 = 7n, \quad n \geq 0,\ a_0 = a_1 = 1.$

Problem 5. Solve $a_n^2 - 2a_{n-1} = 0, \quad n \geq 1,\ a_0 = 2$;(Let $b_n = \log_2 a_n$).

Problem 6. Let a_n denote the nth Fibonacci number, $n \geq 0$. Let $\alpha = (1 + \sqrt{5})/2$. For $n \geq 3$,show that $a_n > \alpha^{n-2}$ and $a_n < \alpha^{n-1}$.

Problem 7. Let a_n be the Fibonacci sequence. Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\sqrt{5}+1}{2}.$$

Problem 8. Find a 3-term recurrence relation for the sequence $a_n = 3^{n+1} - 2 \times 5^n$. Now do the same for $a_n = 3^{n+1} - 2 \times 5^n + n^2$.

Problem 9. Suppose that $a_0 = 0, b_0 = 1$ and that $a_n = a_{n-1} + 2b_{n-1}$, $b_n = -a_{n-1} + 4b_{n-1}$. Find a_n and b_n.

Problem 10. Compute

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

Problem 11. Suppose $a_{n+1} = a_n + a_{n-1}$ and $a_0 = 0$. If a_n converges to some real number as $n \to \infty$. What is a_1?

Problem 12. Let $a_n = \lfloor (5+\sqrt{21})^n \rfloor + 1$, here $\lfloor x \rfloor$ is the floor function, i.e. the largest interger that is less that x. So $a_0 = 2, a_1 = 10, a_2 = 92$.

(1) Let $b_n = (5-\sqrt{21})^n$, then $b_n < 1$. Show that $a_n = (5+\sqrt{21})^n + (5-\sqrt{21})^n$.
(2) Prove that 2^n is a factor of a_n.

Problem 13. Suppose $a_0 = 0, a_1 = 2$ and $a_{n+2} = 4a_{n+1} - 4a_n + n^2 - 5n + 2$. Find a_n and then show that n divides a_n.

Problem 14. A derangement of $\{1, 2, \ldots, n\}$ is a permutation of $\{1, 2, \ldots n\}$ such that nothing is in its right place, i.e. 1 is not in first place, 2 is not in second place(its natural position). For example, $(2, 1, 4, 3)$ is a derangement of $(1, 2, 3, 4)$, but $(1, 3, 4, 2)$ is not. Let d_n denotes the number of derangements of $\{1, 2, 3 \ldots, n\}$.

(1) If $n > 2$, show that d_n satisfies the recurrence relation
$$d_n = (n-1)(d_{n-1} + d_{n-2}), \quad d_2 = 1, d_1 = 0.$$
(2) Show that d_n satisfies
$$d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2}).$$
and $d_n - nd_{n-1} = (-1)^n$.
(3) Show by induction that
$$d_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots + \frac{(-1)^n}{n!}).$$

Problem 15. If $a_0 = 0, a_1 = 1, a_2 = 4$ and $a_3 = 37$ satisfy the recurrence relation $a_{n+2} + ba_{n+1} + ca_n = 0$, where $n \geq 0$ and b, c are constants. Determine b, c and solve for a_n.