Problem 1. Find the generating function of the following sequence

(1) $a_{n+1} - a_n = 3^n, \quad n \geq 0, a_0 = 1;$
(2) $1, 1, 2, 2, 2 \ldots 2, \ldots;$
(3) $a_n = n^3$, i.e. $1, 8, 27, \ldots;$

Problem 2. Find the sequence corresponding to the following generating function

(1) $f(x) = \frac{2}{1-3x} + \frac{1}{1-x}$;

(2) $f(x) = \frac{x}{(1-x)^2}$;

Problem 3. Use the method of generating to solve

$$a_{n+2} = 3a_{n+1} - 2a_n, \quad a_0 = 1, a_1 = 6;$$

Problem 4. Use the method of generating function to solve

$$a_{n+2} = 2a_{n+1} - a_n + 2^n, \quad a_0 = 1, a_1 = 2;$$

Problem 5. For $n = 5^k$, find $f(n)$ if

$$f(n) = 2f(n/5) + 3$$

and $f(1) = 0$.

Problem 6. Find three positive integers a, b, c such that $31 \mid (5a + 7b + 11c)$.

Problem 7. Let a, b be two positive numbers. If $b \mid a$ and $b \mid (a+2)$. Show that $b = 1$ or $b = 2$.

Problem 8. If n is an odd positive integer, show that $8 \mid (n^2 - 1)$.

Problem 9. Determine the quotient q and remainder r for each of the following, where a is the dividend and b is the divisor

(1) a=23,b=7;
(2) a=-115,b=12;
(3) a=0,b=42;

Problem 10. Write each of the following (base 10) integers in base 2 and base 3:

$$137, \quad 6243, \quad 12345$$

Problem 11. For what base do we find that $251 + 445 = 1026$?

Problem 12. Find all $n \in \mathbf{Z}^{+}$ such that n divides $5n + 18$.

Problem 13. Compute (Express the answer in the same base)
(1) $(101001)_2 + (1011111)_2$;
(2) $(122111)_3 + (1022100)_3$;

Problem 14. Suppose b_n is determined by the following recurrence relation

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \ldots b_{n-1} b_1 + b_n b_0.$$

with $b_0 = 1$.
(1) Denote $f(x)$ the generating function of b_n. Show that

$$f(x) - b_0 = x(f(x))^2$$

(2) Compute $f(x)$;
(3) Expand $f(x)$ and find b_n.

Problem 15. Use a different method to show that there are infinitly many prime numbers.