HOMEWORK 4

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly(It is up to you whether to print it double sided or not). Due on Friday, Feb 10

Problem 1. Show that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Problem 2. Show that
$$\sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2}$$
by taking the second derivative of the expansion of $(1 + x)^n$.

Problem 3. We are going to use two methods to prove the following

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}.$$

(1) (Double counting) Consider the following problem: how many ways are there to choose k from n objects. We can split n objects into two halfs, one with m objects and the other one with $n-m$. Complete the argument to prove the identity.

(2) (Binomial theorem) Prove the identity by considering the coefficient of x^k in $(1+x)^m(1+x)^{n-m} = (1+x)^n$.

(3) Use the identity you just proved to show that

$$\sum_{j=0}^{m} \binom{m}{j}^2 = \binom{2m}{m}.$$

Problem 4. Determine the number of nonnegative integer solutions to the following equations

$$x_1 + x_2 + x_3 = 6.$$

What if the equation is

$$x_1 + x_2 + x_3 \leq 6.$$

Problem 5. Find the coefficient of x^3 in $(x + 2/x^3)^9$.

Problem 6. A die is tossed ten times and the sequence of the outcomes is observed.

(1) How many different sequences are possible?
(2) How many of these sequences contain a row of exactly three 4's?
(3) How many of these sequences contain at most 1's?
(4) How many of these sequences have sum equal to 13?

Problem 7. Sixteen people are to be seate at two circular tables, one of which seats 10 while the other seats six. How many different seating arrangements are possible?

Problem 8. A committee of 15-nine women and six men-is to be seated at a circular talbe. In how many ways can the seats be assigned so that no two men are seated next to each other?

Problem 9. How many positive integers n can be formed using the digits 3,4,4,5,5,6,7 if we want n to exceed 5000000?

Problem 10. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

Problem 11. Four numbers are selected from the following list of numbers: -5, -4, -3, -2, -1, 1, 2, 3, 4. In how many ways can the selections be made so that the product of the four numbers is positive (the numbers are distinct)?

Problem 12. In how many ways can 11 identical balls be painted so that three are brown, three are white and five are black? What if the balss are distinct?

Problem 13. In how many ways can 17 be written as a sum of 2's and 3's if the order of the summands is (i) not relevant (ii) relevant?

Problem 14. In how many ways can one toss 100 dice so that all 6 types of face will be showing?

Problem 15. Show that

$$\sum_{k=0}^{n} k^3 = \left(\frac{(n+1)n}{2}\right)^2$$

by using the following fact

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

for any r, n with $1 \leq r \leq n$.