HOMEWORK 7

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly(It is up to you whether to print it double sided or not). Due on Friday, Mar 2

Problem 1. In how many ways can the integers $1, 2, \ldots, 10$ be arranged in a line so that no even integer is in its natural position?

Problem 2. Let d_n denote the number of derangements of $1, 2, \ldots, n$. Show by double counting that

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_k.$$

Problem 3. Let $A \subset \{1, 2, \ldots, 25\}$ where $|A| = 9$. For any subset B of A, let s_B denote the sum of the elements in B. Prove that there are distinct subsets C, D of A such that $|C| = |D| = 5$ and $s_C = s_D$.

Problem 4. For any $k \in \mathbf{Z}^+$, show that there exists a positive integer n such that $k \mid n$ and the only digits in n are 0's and 1's.

Problem 5. How many times must we roll a single die in order to get the same face at least n times.

Problem 6. Prove that if we select 101 integers from the set $S = \{1, 2, \ldots, 200\}$, there exist m, n in the selection such that $\gcd(m, n) = 1$.

Problem 7. Let $S \subset \mathbf{Z}^+$ and $|S| \geq 3$. Show that there exist distinct $x, y \in S$ such that $x + y$ is even.

Problem 8. Seven distinct objects are to be put into three distinct boxes. How many ways are there such that no box is empty and each box contains at most 3 objects.

Problem 9. Determine the number of (staircase) paths in the $x - y$ plane from $(0, 0)$ to $(8, 4)$ where each such path is made up of individual steps going one unit to the right or one unit upward in the following case

(1) There is no other constraint.
(2) The paths must pass through $(3, 2)$ and $(4, 2)$.
(3) The paths must pass though at least one of $(3, 2) - (4, 2)$, $(2, 1) - (3, 1)$, $(4, 3) - (5, 3)$.

Problem 10. Compute $\phi(2^n p)$ where p is an odd prime number.

Problem 11. In how many ways can we arrange the integer $1, 2 \ldots, 8$ in a line so that there are no occurrences of the patterns $12, 23, \ldots, 78, 81$?

Problem 12. Determine the number of positive integers x where $x \leq 999999$ and the sum of the digits in x equals 31.

Problem 13. If $\{x_1, x_2 \ldots, x_7\} \subset \mathbf{Z}^+$, show that for some $i \neq j$, either $x_i + x_j$ or $x_i - x_j$ is divisible by 10.

Problem 14. Let n be a positive integer, and $i_1, i_2, \ldots, i_n$ be a permutation of $1, 2 \ldots, n$, show that $(1 - i_1)(2 - i_2) \cdots (n - i_n)$ is an even integer.

Problem 15. Let m be a positive odd integer. Show that there exists a positive integer n such that $m \mid 2^n - 1$.