HOMEWORK 3

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly. Due on Monday, Feb 18.

Problem 1. Show that if $P(A|B) = 1$, then $P(B^c|A^c) = 1$.

Problem 2. An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly ace. (Use $|E|/|S|$ instead of conditional probability)

Problem 3. The king comes from a family of 2 children. What is the probability that the other child is his sister?

Problem 4. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade given that the second and third cards are spades.

Problem 5. Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability $P(E)/(P(E) + P(F))$.

Problem 6. Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads and let C be the event that in both tosses the coin lands on the same side. Show that A, B, C are pairwise independent but not independent.

Problem 7. Let Q_n denote the probability that no run of 3 consecutive heads appears in n tosses of a fair coin. Show that

$$Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}$$
$$Q_0 = Q_1 = Q_2 = 1$$

and find Q_8 (Hint: condition on when the first tail occurs).

Problem 8. The Ballot Problem. In an election, candidate A and B receives n votes and candidate B receives m votes, where $n > m$. Assuming all of the $(n+m)!/(n!m!)$ orderings of the votes are equally likely, let $P_{n,m}$ denote the probability that A is always ahead in the counting of the votes

(1) Compute $P_{2,1}, P_{3,1}, P_{3,2}, P_{4,1}, P_{4,2}, P_{4,3}$.
(2) Find $P_{n,1}, P_{n,2}$.
(3) Based on the above results, conjecture the value of $P_{n,m}$.
(4) Derive a recursion for $P_{n,m}$ in terms of $P_{n-1,m}$ and $P_{n,m-1}$ by conditioning on who receives the last vote.
(5) Verify the conjecture in (3) by plugging into (4).

Problem 9. Throw 3 dice one by one. What is the probability that 3 faces are in strictly increasing order?

Problem 10. There is one amoeba in a pond. After every minute the amoeba may die, stay the same, split into two or split into three with equal probability. All its offsprings, if it has any, will behave the same (and independent of other amoebas). What is the probability the amoeba population will die out?

Problem 11. Let $A \subset B$. Express the following probabilities as simply as possible.

(1) $P(A|B)$
(2) $P(A|B^c)$
(3) $P(B|A)$
(4) $P(B|A^c)$

Problem 12. Independent trials of throwing an unfair die. The probability of getting an even number is 2/3 while the probability of getting an odd number is 1/3. What is the probability that a run of 3 consecutive even numbers occurs before a run of 4 consecutive failures?

Problem 13. (1) A company is holding a dinner for working mothers with at least one son. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?
(2) Ms. Parker is know to have two children. If you see her walking with one of her children and that child is a boy, what is the probability that both children are boys?

Problem 14. If A flips $n+1$ and B flips n fair coins, show that the probability that A gets more heads than B is $\frac{1}{2}$.

Problem 15. (1) A box contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that, with probabilit $n/(n+m)$, they are all white. (Hint: Imagine that the experiment continues until all the balls are removed, show that the event is actually equivalent to the event that the last ball is white).
(2) A box contains r red, b blue and g green balls. They are removed from the box in a random order. What is the probability that red balls are the first to empty out from the box. (Hint: Denote $P(R)$ be the probability that red balls are the first to be gone and $P(RBG)$ be the probability that red balls are gone first and then blue, then green. We have $P(R) = P(RBG) + P(RGB)$. Compute the right by first conditioning on the last ball to be removed)