HOMEWORK 4

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly. Due on Monday, March 4.

Problem 1. Suppose that the distribution function of X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{4}, & 0 \le x < 1 \\ \frac{1}{2} + \frac{x-1}{4}, & 1 \le x < 2 \\ \frac{11}{12}, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$

(1) Find $P(X = i), i = 1, 2, 3$.
(2) Find $P(0.5 < X < 1.5)$.

Problem 2. Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

Problem 3. If $E[X] = 1$ and $\text{Var}(X) = 5$, find

(1) $E[(2+X)^2]$
(2) $\text{Var}(4+3X)$

Problem 4. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is 0.01, what is the (approximate) probability that you will win a prize

(1) at least once?
(2) exactly once?
(3) at least twice?

Problem 5. Let X be a Poisson random variable with parameter λ. Show that

$$P(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda})$$

Problem 6. What is the expected number of cards that need to be turned over in a regular 52-card deck in order to see the first ace?

Problem 7. Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size n flies overhead.

Problem 8. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(1) What is the value of c?
(2) What is the cumulative distribution function of X?
(3) Find the expectation of X.

Problem 9. If X is uniformly distributed over $(-1, 1)$, find

(1) $P(|X| > 0.5)$
(2) the density function of the random variable $|X|$.

Problem 10. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

Problem 11. Let X be a normal random variable with mean 12 and variance 4. Find the value c such that $P(X > c) = 0.1$.

Problem 12. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(1) $P(X > 5)$
(2) $P(4 < X < 16)$
(3) $P(X < 8)$
(4) $P(X < 20)$
(5) $P(X > 16)$

Problem 13. There are N distinct types of coupons in cereal boxes and each type, independent of prior selections, is equally likely to be in a box. What is the expected number of distinct coupon types if a child has collected n coupons?

Problem 14. Show that X is Poisson random variable with parameter λ, then

$$E[X^n] = \lambda E[(X+1)^{n-1}].$$

Use this result to compute $E[X^3]$.

Problem 15. A box contains $m + n$ balls, numbered $1, 2, \cdots, n + m$. A set of size n is drawn. If we let X denote the number of balls drawn having numbers that exceed each of the numbers of those remaining, compute the probability mass function of X.