HOMEWORK 5

Instruction: Please complete the first 10 problems. The last 5 problems are for you to practice, they are candidates for tests. Please print the homework and work on it directly. Due on Monday, March 25.

Problem 1. Two fair dice are rolled. Find the joint probability mass function of X and Y when X is the value on the first die and Y is the larger of the two values.

Problem 2. Suppose that 3 balls are chosen without replacement from a box consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(1) X_1, X_2;
(2) X_1, X_2, X_3.

Problem 3. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}(x^2 + \frac{xy}{2}), \quad 0 < x < 1, 0 < y < 2.$$

(1) Verify that this is indeed a joint density function;
(2) Compute the density function of X;
(3) Find $P(X > Y)$;
(4) Find $P(Y > 1/2 | X < 1/2)$;
(5) Find $E[X]$.

Problem 4. The number of people that enter a drugstore in a given hour is Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

Problem 5. Show that

$$f(x, y) = \frac{1}{x}, \quad 0 < y < x < 1$$

is a joint density function. Assuming that f is the joint density function of X and Y, find

(1) the marginal density of X;
(2) the marginal density of Y;
(3) $E[X]$.

Problem 6. The joint probability mass function of X and Y is given by

$$p(1,1) = \frac{1}{8}, \quad ,p(1,2) = \frac{1}{4}, \quad p(2,1) = \frac{1}{8}, \quad p(2,2) = \frac{1}{2}.$$

(1) Compute the conditional mass function of X given $Y = i, i = 1, 2$;
(2) Are X and Y independent?
(3) Compute $P(XY \leq 3), P(X + Y > 2), P(X/Y) > 1$.

Problem 7. If $X_1, X_2, \cdots, X_5$ are independent and identically distributed exponential random variables with parameter λ, compute

(1) $P(\min(X_1, \cdots, X_5) \leq a)$;
(2) $P(\max(X_1, \cdots, X_5) \leq a)$.

Problem 8. If X and Y are independent and identically distributed uniform random variables on $(0, 1)$, compute the joint density of

(1) $U = X + Y$, $V = X/Y$;
(2) $U = X$, $V = X/Y$;
(3) $U = X + Y$, $V = X/(X + Y)$.

Problem 9. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3?

Problem 10. Suppose A, B, C are independent random variables, each being uniformly distributed over $(0, 1)$.

(1) What is the joint cumulative distribution function of A, B, C?
(2) What is the probability that all of the roots of the equation $Ax^2 + Bx + C = 0$ are real?

Problem 11. If X and Y are independent random variables both uniformly distributed over $(0,1)$, find the joint density function of $R = \sqrt{X^2+Y^2}$ and $\Theta = \tan^{-1}(Y/X)$.

Problem 12. Let U denote a random variable uniformly distributed over $(0,1)$. Compute the conditional distribution of U given that

(1) $U > a$;
(2) $U < a$.

Problem 13. Show that the median of a sample of size $2n+1$ from a uniform distribution on $(0,1)$ has a beta distribution with parameter $(n+1, n+1)$.

Problem 14. Let $X_1, \cdots, X_n$ be independent uniform $(0,1)$ random variables. Let $R = X_{(n)} - X_{(1)}$ denote the range and $M = (X_{(n)} + X_{(1)})/2$ the midrange. Compute the joint density function of R and M.

Problem 15. The joint probability mass function of the random variables X, Y, Z is

$$p(1,2,3) = p(2,1,1) = p(2,2,1) = p(2,3,2) = \frac{1}{4}.$$

Find

(1) $E[XYZ]$;
(2) $E[XY + XZ + YZ]$.