HOMEWORK 6

Instruction: Please complete all the problems. The first 10 problems are also candidates for test 3. Please print the homework and work on it directly. Due on Friday, April 26.

Problem 1. Show that if X and Y are identically distributed (not necessarily independent), then

$$\mathrm{Cov}(X+Y, X-Y) = 0.$$

Problem 2. Show that $Y = a + bX$, then

$$\rho(X,Y) = \begin{cases} 1, & b > 0 \\ -1 & b < 0. \end{cases}$$

Problem 3. If $X_1, \cdots, X_n$ are independent and identically distributed random variables having uniform distribution over $(0,1)$, find

(1) $E[\max(X_1, \cdots, X_n)]$;
(2) $E[\min(X_1, \cdots, X_n)]$;

Problem 4. Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X, Y)$.

Problem 5. A fair die is successively rolled. Let X and Y denote the number of rolls necessary to obtain a 6 and 5, respectively. Find

(1) $E[X]$;
(2) $E[X|Y = 1]$;
(3) $E[X|Y = 5]$.

Problem 6. Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each choose his target at random, independent of the others. If each hunter independently hit his target with probability 0.6, compute the expected number of ducks that are hit. Assume that the number of ducks in a flock is a Poisson random variable with mean 6.

Problem 7. The moment generating function of X is given by $M_X(t) = \exp(2e^t - 2)$ and that of Y by $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$. If X and Y are independent, what are

(1) $P(X + Y = 2)$?
(2) $P(XY = 0)$?
(3) $E[XY]$?

Problem 8. If X is a standard normal random variable, what is $\text{Cov}(X, X^2)$ (Hint: use moment generating function to compute $E[X^3]$)?

Problem 9. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 2e^{-2x}/x, & 0 \le x < \infty, 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$

Compute $\mathrm{Cov}(X,Y)$.

Problem 10. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty.$$

Compute $E[X^3|Y = y]$.

Problem 11. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where σ^2 is a known positive number. Show that the maximum likelihood estimator for μ is $\overline{X}$.

Problem 12. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where μ is a known positive number. Show that the maximum likelihood estimator for σ^2 is

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

Problem 13. A random sample $X_1, \cdots, X_n$ of size n is taken from a Poisson distribution with a mean of λ,

(1) Show that the maximum likelihood estimator for λ is $\widehat{\lambda} = \overline{X}$.
(2) Let X equal the number of flaws per 100 feet of a used computer tape. Assume that X has a Poisson distribution with a mean of λ. If 40 observations of X yielded 5 zeros, 7 ones, 12 twos, 9 threes, 5 fours, 1 five, and 1 six, find the maximum likelihood estimate of λ.

Problem 14. Let $X_1, \cdots, X_n$ be a ramdon sample from distributions with the given probability density function. In each case, find the maximum likelihood estimator of $\widehat{\theta}$.

(1) $f(x;\theta) = (1/\theta^2)xe^{-x/\theta}$, $x > 0, \theta > 0$.
(2) $f(x;\theta) = (1/2\theta^3)x^2e^{-x/\theta}$, $x > 0, \theta > 0$.

Problem 15. Let $X_1, \cdots, X_n$ be a random sample from binomial distribution $B(1, p)$. Let $Y = X_1 + \cdots + X_n$.

(1) Show that $\overline{X} = Y/n$ is an unbiased estimator of p.
(2) Show that $\mathrm{Var}(X) = p(1-p)/n$.
(3) Show that $E[\overline{X}(1-\overline{X})/n] = (n-1)p(1-p)/n^2$.
(4) Find the value of c such that $c\overline{X}(1-\overline{X})$ is an unbiased estimator of $\mathrm{Var}(\overline{X}) = p(1-p)/n$.

Problem 16. A random sample of size 16 from the normal distribution $N(\mu, 25)$ yields $\overline{x} = 73.8$. Find a 95% confidence interval for μ.

Problem 17. Assume that the yield per acre for a particular variety of soybean is $N(\mu, \sigma^2)$. For a random sample of $n = 5$ plots, the yields in bushels per acre were $37.4, 48.8, 46.9, 55.0, 44.0$.

(1) Give a point estimate for μ.
(2) Find a 90% confidence interval for μ.

Problem 18. Let X equal the length of a certain species of fish caught in the springtime. A random sample of $n = 13$ observations of X is

$$13.1 \quad 5.1 \quad 18.0 \quad 8.7 \quad 16.5 \quad 9.8 \quad 6.8 \quad 12.0 \quad 17.8 \quad 25.4 \quad 19.2 \quad 15.8 \quad 23.0$$

(1) Give a point estimation of the standard deviation σ of this species of fish.
(2) Find a 95% confidence interval for σ.

Problem 19. A random sample of $n = 9$ wheels of cheese yielded the following weights in pounds, assumed to be $N(\mu, \sigma^2)$:

$$21.50 \quad 18.95 \quad 18.55 \quad 19.40 \quad 19.15 \quad 22.35 \quad 22.90 \quad 22.20 \quad 23.10$$

(1) Give a point estimation of σ.
(2) Find a 95% confidence interval for σ.